Dimensional Properties of Subsystems of Toral Endomorphisms

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April 2026

Abstract

Let $A : \mathbb{T}^n \to \mathbb{T}^n$ be a toral endomorphism. In this paper, we investigate the dimensional properties of subsystems of $(A, \mathbb{T}^n)$. Specifically, we attempt to address the following questions: Under what conditions is the Hausdorff dimension equal to the Minkowski dimension ($\dim_H X = \dim_M X$) for all subsystems $X \subseteq \mathbb{T}^n$? In cases where these dimensions diverge, what characterizes such subsystems, and what is their density within $(A, \mathbb{T}^n)$? How do these dimensional properties relate to the properties of the matrix A?

Introduction

For a set E in a metric space, we define the s-dimensional Hausdorff measure as:

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : E \subseteq \bigcup_{i=1}^{\infty} U_i, \operatorname{diam} U_i < \delta \right\}$$

The Hausdorff dimension is then given by:

$$\dim_H E = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\}$$

Now, define

$$N(E, \epsilon) = \min \left\{ k \in \mathbb{N} : E \subseteq \bigcup_{i=1}^{k} B(x_i, \epsilon),\ x_1, \ldots, x_k \in \mathbb{R}^n \right\}$$

The upper and lower Minkowski dimensions are then defined as:

$$\overline{\dim}_M E = \limsup_{\epsilon \to 0} \frac{\log N(E, \epsilon)}{\log(1/\epsilon)}, \quad \underline{\dim}_M E = \liminf_{\epsilon \to 0} \frac{\log N(E, \epsilon)}{\log(1/\epsilon)}$$

When these limits coincide, we refer to the common value as $\dim_M E$.

We always have the inequality chain: $\dim_H E \leq \underline{\dim}_M E \leq \overline{\dim}_M E$. There exist many examples where these inequalities are strong (see, for instance, Bishop and Peres, *Fractal Sets in Probability and Analysis*). A standard example is the set

$$E = \{0\} \cup \{1/n : n \in \mathbb{N}\}$$

for which it is well-known that $\dim_H E = 0$, while $\dim_M E = 1/2$.

On the other hand, let us consider the map $A : \mathbb{T}^1 \to \mathbb{T}^1$ given by $x \mapsto nx \pmod 1$ for $n \in \mathbb{Z}, |n| > 1$. For any invariant subsystem X, it is a classical result that $\dim_H X = \dim_M X$.

Moving to $\mathbb{T}^2$, the behavior becomes more complex. For a diagonal matrix $A = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$, if $n = m$, the system behaves essentially like the one-dimensional case and the dimensions coincide. However, if $n \neq m$, the dimensions can differ. Take the Bedford–McMullen carpets:

$$K = \left\{ \left(\sum_{i=1}^{\infty} \frac{x_i}{n^i}, \sum_{i=1}^{\infty} \frac{y_i}{m^i} \right) : (x_i, y_i) \in D \right\}$$

where $D \subset \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$. A prominent class of examples where dimensions differ is obtained from certain (but not all) subsets D (For example, $\{(0,0),(1,1),(2,0)\} \subseteq \{0,1,2\} \times \{0,1\}$). In this paper, we broaden this dimensional analysis to some other integer two-dimensional matrices.

Organization of the Paper

This paper is structured as follows:

Chapter 0:Bowen's Markov Partitions: An Overview

This chapter gives a brief overview of Bowen's work on Markov partitions, which becomes useful for chapter 2 and chapter 4.

Chapter 1: The Invertible Case in $M_{2\times 2}(\mathbb{Z})$

In the first chapter, we treat the case of an invertible matrix on the torus ($\det(B) = \pm 1$). We use Markov partitions as defined and constructed by Bowen to obtain a factor map from a symbolic space to the torus. By applying the mass distribution principle, the Shannon–McMillan–Breiman Theorem, and the variational principle, we establish the connection between topological entropy and Minkowski dimension to obtain $\dim_M(X) = \dim_H(X)$.

Chapter 2: Matrices with Equal Eigenvalues in $M_{2\times 2}(\mathbb{Z})$

In the second chapter, we treat the case of a matrix with equal eigenvalues. We utilize a theorem by Falconer (1989) regarding quasi-self-similar sets. While a matrix with equal eigenvalues and a non-trivial Jordan block does not meet these exact conditions due to polynomial growth, it meets conditions sufficiently close to utilize a slight variation of this theorem.

Chapter 3: Mixed Eigenvalues and Non-Invertibility in $M_{2\times 2}(\mathbb{Z})$

In the third chapter, we treat the case of a matrix B with eigenvalues $0 < \lambda_1 < 1 < \lambda_2$, and $|\lambda_1 \lambda_2| \neq 1$. Here, we cannot naively create a Markov partition because utilizing Bowen's work directly requires the map to be invertible. We attempt to create a Markov partition for the invertible lift of our space. In addition, we draw conclusions from the dynamic shift space $\{0, \ldots, n\}^{\mathbb{Z}} \times \{0, \ldots, m\}^{\mathbb{N}}$, which shares significant structural similarities to our space. Finally, we discuss the cases of matrices with eigenvalues $1 < \lambda_1, \lambda_2$ and potential extensions to higher-dimensional tori.

1 Bowen's Markov Partitions: An Overview

In this section, we provide an overview of the classical construction of Markov partitions following Bowen [1, pp. 55–66]. Let $f : M \to M$ be a diffeomorphism of a compact C^∞ Riemannian manifold M. Then the derivative of f can be considered a map $df : TM \to TM$ where $TM = \bigcup_{x \in M} T_x M$ is the tangent bundle of M and $df_x : T_x M \to T_{f(x)} M$.

Definition 1.1 (Hyperbolic Set)**.** A closed subset $\Lambda \subset M$ is **hyperbolic** if $f(\Lambda) = \Lambda$ and each tangent space $T_x M$ with $x \in \Lambda$ can be written as a direct sum $T_x M = E^u_x \oplus E^s_x$ of subspaces so that:

(a) $Df(E^s_x) = E^s_{f(x)}$, $Df(E^u_x) = E^u_{f(x)}$;

(b) there exist constants $c > 0$ and $\lambda \in (0, 1)$ so that $\|Df^n(v)\| \leq c\lambda^n \|v\|$ when $v \in E^s_x, n \geq 0$ and $\|Df^{-n}(v)\| \leq c\lambda^n \|v\|$ when $v \in E^u_x, n \geq 0$;

(c) E^s_x, E^u_x vary continuously with x.

Definition 1.2 (Anosov Diffeomorphism)**.** f is an **Anosov diffeomorphism** if M is a hyperbolic set for f.

Definition 1.3 (Non-wandering Set)**.** A point $x \in M$ is **non-wandering** if $U \cap \bigcup_{n>0} f^n U \neq \emptyset$ for every neighborhood U of x. The set $\Omega(f)$ of all non-wandering points is closed and f-invariant.

Definition 1.4 (Axiom A)**.** f satisfies **Axiom A** if:

1. $\Omega(f)$ is hyperbolic.
2. The periodic points of f are dense in $\Omega(f)$.

Definition 1.5. Let Λ be a hyperbolic set. For a sufficiently small $\epsilon > 0$ and any point $x \in \Lambda$, we define:

$$\begin{aligned}
\text{Local stable manifold:} \quad & W^s_\epsilon(x) = \{y \in M : d(f^n x, f^n y) \leq \epsilon \text{ for all } n \geq 0\} \\
\text{Local unstable manifold:} \quad & W^u_\epsilon(x) = \{y \in M : d(f^{-n} x, f^{-n} y) \leq \epsilon \text{ for all } n \geq 0\} \\
\text{Global stable manifold:} \quad & W^s(x) = \{y \in M : d(f^n x, f^n y) \to 0 \text{ as } n \to \infty\} \\
\text{Global unstable manifold:} \quad & W^u(x) = \{y \in M : d(f^{-n} x, f^{-n} y) \to 0 \text{ as } n \to \infty\}
\end{aligned}$$

where d is the adapted Riemannian metric on the manifold M.

Theorem 1.6 (Anosov's Closing Lemma)**.** *If f is an Anosov diffeomorphism, then f satisfies Axiom A.*

Theorem 1.7 (Spectral Decomposition)**.** *Let f satisfy Axiom A. One can write $\Omega(f) = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_s$ where the Ω_i are pairwise disjoint closed sets with:*

(a) $f(\Omega_i) = \Omega_i$ and $f|_{\Omega_i}$ is topologically transitive;

(b) $\Omega_i = X_{1,i} \cup \cdots \cup X_{n_i,i}$ with the $X_{j,i}$'s pairwise disjoint closed sets, $f(X_{j,i}) = X_{j+1,i}$ (where $X_{n_i+1,i} = X_{1,i}$) and $f^{n_i}|_{X_{j,i}}$ is topologically mixing.

1.1 Canonical Coordinates

Theorem 1.8 (Canonical Coordinates)**.** *Suppose f satisfies Axiom A. For any small $\epsilon > 0$ there is a $\delta > 0$ so that $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ consists of a single point, denoted by $[x, y]$, whenever $x, y \in \Omega(f)$ and $d(x, y) \leq \delta$. Furthermore, $[x, y] \in \Omega(f)$, and the bracket map $[\cdot, \cdot] : \{(x, y) \in \Omega(f) \times \Omega(f) : d(x, y) \leq \delta\} \to \Omega(f)$ is continuous.*

1.2 Markov Partitions

Let Λ be a hyperbolic set. A **rectangle** is a set $T \subset \Lambda$ such that T has small diameter and $[x, y] \in T$ whenever $x, y \in T$. For $x \in T$ let

$$W^s(x, T) = W^s_\epsilon(x) \cap T$$
$$W^u(x, T) = W^u_\epsilon(x) \cap T$$

where ϵ is the diameter of T. A rectangle T is **proper** if $T = \overline{\text{int } T}$ (as a subset of Λ).

A **Markov partition** of Λ is a finite collection $\mathcal{R} = \{R_1, \ldots, R_m\}$ of proper rectangles such that:

(a) $\Lambda = \bigcup_{i=1}^m R_i$;

(b) $\text{int } R_i \cap \text{int } R_j = \emptyset$ for $i \neq j$;

(c) if $x \in \text{int } R_i$ and $f(x) \in \text{int } R_j$, then $f(W^s(x, R_i)) \subset W^s(f(x), R_j)$ and $f(W^u(x, R_i)) \supset W^u(f(x), R_j)$.

Theorem 1.9. *Every basic set Ω_s has Markov partitions $\mathcal{R} = \{R_1, \ldots, R_m\}$ of arbitrarily small diameter.*

Definition 1.10 (Transition Matrix)**.** Let $\mathcal{R} = \{R_1, \ldots, R_m\}$ be a Markov partition for a hyperbolic set Λ. The **transition matrix** $A(\mathcal{R})$ is the $m \times m$ matrix defined by:

$$A_{ij} = \begin{cases} 1 & \text{if int } R_i \cap f^{-1}(\text{int } R_j) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Definition 1.11 (Symbolic Space Σ_A)**.** Given a transition matrix A, the **topological Markov chain** (or subshift of finite type) (Σ_A, σ) is defined by the space of sequences:

$$\Sigma_A = \{\mathbf{x} = (x_k)_{k \in \mathbb{Z}} : x_k \in \{1, \ldots, m\} \text{ and } A_{x_k x_{k+1}} = 1 \text{ for all } k \in \mathbb{Z}\} \tag{2}$$

equipped with the product topology and the **shift map** $\sigma : \Sigma_A \to \Sigma_A$ defined by $(\sigma x)_k = x_{k+1}$.

Theorem 1.12. *Let $\mathcal{R}$ be a Markov partition for a basic set Ω_s. The map $\pi : \Sigma_A \to \Omega_s$ defined by*

$$\pi(x) = \bigcap_{k \in \mathbb{Z}} f^{-k} R_{x_k}$$

is a continuous surjection such that $\pi \circ \sigma = f \circ \pi$. Furthermore, π is one-to-one on $\pi^{-1}(\Omega_s \setminus \bigcup_{k \in \mathbb{Z}} f^k \partial\mathcal{R})$.

2 The Hyperbolic Invertible Case

Let $B : \mathbb{T}^2 \to \mathbb{T}^2$ be a toral automorphism with eigenvalues $\lambda_1 = \lambda_2^{-1}$ and $\lambda_1 > 1$. We utilize Bowen's framework to construct a Markov partition of the space in order to analyze its invariant subsystems.

2.1 Key Dynamical Observations

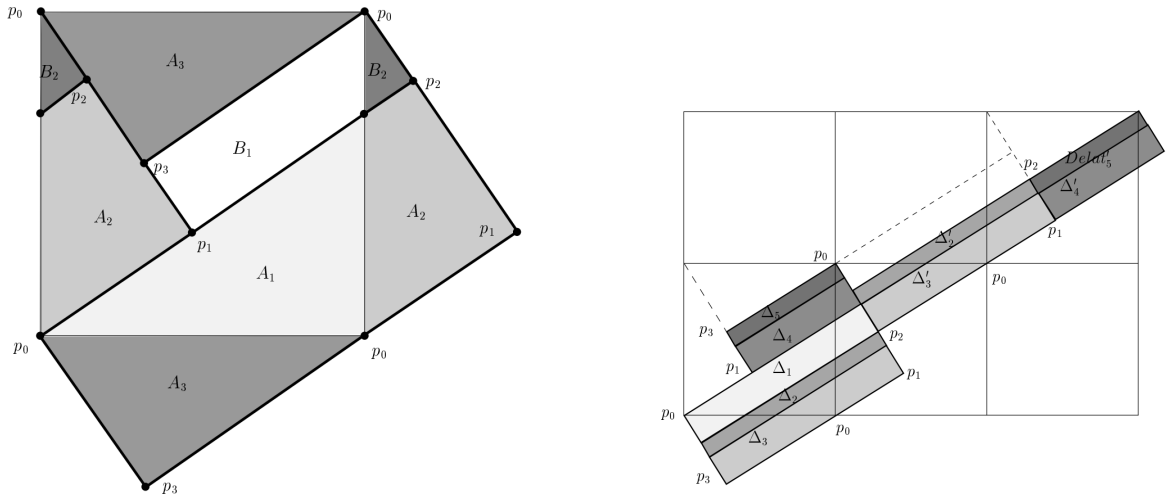
Claim 2.1. *The map B is an Anosov diffeomorphism.*

Proof. We define the stable and unstable subspaces at each point to be those generated by the respective contracting and expanding eigenvectors of B, setting λ to be λ_1^{-1}. □

Claim 2.2. *The map B is topologically mixing.*

Proof. Let U be a nonempty open set in $\mathbb{T}^2$. Consider a line segment $L \subset U$ aligned with the expanding unstable direction. Since B is a hyperbolic toral automorphism, its eigenvalues are irrational, implying that the unstable manifold has an irrational slope. By standard results in structural dynamics, the projections of these long unstable segments are dense in the torus, ensuring that $B^n(U)$ eventually intersects any other open set for sufficiently large n. □

Lemma 2.3. *The torus $\mathbb{T}^2$ consists of a single basic set (itself), where stable and unstable manifolds are linear segments along the eigen-directions, and canonical coordinates are unique intersections within small local domains. Rectangles are parallelograms and are uniformly proper.*



(a) Brin-Stuck partition.

(b) Explicit Markov partition geometry.

Figure 1: A Markov partition of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ obtained from Brin and Stuck [3].

We obtain a partition $\mathbb{T}^2 = \bigcup_{i=1}^{k} R_i$ and a finite-to-one semi-conjugacy $\phi : \Sigma_A \to \mathbb{T}^2$. We fix a fundamental domain of the torus that leaves each partition parallelogram within a single connected component. For any word $(a_{-m}, \ldots, a_n) \in L_{n+m+1}(\Sigma_A)$, we define:

$$\phi(a_{-m}, \ldots, a_n) := \bigcap_{k=-m}^{n} B^{-k}(R_{a_k})$$

Let us define the collection of these cylinder regions as:

$$C_{m,n} := \{\phi(a_{-m}, \ldots, a_n) \mid (a_{-m}, \ldots, a_n) \in L_{n+m+1}(\Sigma_A)\}$$

2.2 Properties of the Markov Geometric Cover

Claim 2.4. *For every structural word $(a_{-m}, \ldots, a_n)$, the set $\phi(a_{-m}, \ldots, a_n)$ is a single, connected parallelogram in the chosen fundamental domain of $\mathbb{T}^2$.*

Proof. The intersection $\bigcap_{k=-m}^{n} B^{-k}(R_{a_k})$ lifted to $\mathbb{R}^2$ is a finite intersection of convex parallelograms whose boundaries are uniformly aligned with the eigen-directions. Since it is entirely contained within the proper local domain R_{a_0}, it projects down as a single connected region. □

Claim 2.5. *The collection $C_{m,n}$ forms a proper geometric covering of $\mathbb{T}^2$.*

Proof. Since B is invertible, the collection $\{B^k(R_1), \ldots, B^k(R_k)\}$ covers $\mathbb{T}^2$ for any fixed $k \in \mathbb{Z}$. The collection $C_{m,n}$ is the join of $m + n + 1$ such covers, preserving the global coverage properties. □

Let a and b denote the minimum and maximum side lengths across all initial rectangles R_i, respectively.

Claim 2.6. *For every admissible word $(a_{-m}, \ldots, a_n)$, the geometric rectangle $\phi(a_{-m}, \ldots, a_n)$ has component side lengths bounded below by $a\lambda_1^{-n}$ and above by $b\lambda_1^{-m}$.*

Proof. The boundaries of the partition elements are strictly aligned with the stable and unstable dynamics. Iteration under B^k yields dynamic scaling along these axes precisely by factors of λ_1 and λ_1^{-1}. □

Consequently, $C_{n,n}$ is a covering of $\mathbb{T}^2$ by small parallelograms whose widths and lengths are bounded within $[a\lambda_1^{-n}, b\lambda_1^{-n}]$, intersecting only at boundaries.

Let Y be an invariant subsystem of $(\mathbb{T}^2, B)$. We define $\widetilde{N}(Y, n)$ to be the exact number of parallelograms in $\mathcal{B}_{n,n}$ that intersect Y.

2.3 Minkowski Dimension Consistency

Proposition 2.7. *There exist global geometric constants $c_1, c_2 > 0$ independent of n such that:*

$$c_1 N(Y, \lambda_1^{-n}) \leq \widetilde{N}(Y, n) \leq c_2 N(Y, \lambda_1^{-n}) \quad \forall n \in \mathbb{N} \tag{3}$$

Proof. By Claim 2.5, every element $R \in \mathcal{C}_{n,n}$ has side lengths bounded by $a\lambda_1^{-n}$ and $b\lambda_1^{-n}$. Since the interior angles of these parallelograms are fixed uniformly by the global eigenvectors of B, the shape orientation is stable across scales. Thus, there exist geometric constants $K_{\min}, K_{\max} > 0$ such that $\operatorname{diam}(R) \leq K_{\max}\lambda_1^{-n}$ and $\operatorname{Area}(R) \geq K_{\min}\lambda_1^{-2n}$.

For the lower bound, let $\mathcal{R}_Y \subseteq \mathcal{C}_{n,n}$ be the elements intersecting Y, so $|\mathcal{R}_Y| = \widetilde{N}(Y, n)$. Because $\operatorname{diam}(R) \leq K_{\max}\lambda_1^{-n}$, each element can be covered by a bounded number t_1 of standard Euclidean balls of radius λ_1^{-n}. Thus, Y can be covered by $t_1\widetilde{N}(Y, n)$ standard balls, implying $N(Y, \lambda_1^{-n}) \leq t_1\widetilde{N}(Y, n)$, yielding $c_1 = 1/t_1$.

For the upper bound, consider an optimal cover of Y by $N(Y, \lambda_1^{-n})$ standard balls of radius λ_1^{-n}. Any parallelogram $R \in \mathcal{C}_{n,n}$ intersecting a ball $B(x, \lambda_1^{-n})$ is contained within $B(x, (1 + K_{\max})\lambda_1^{-n})$, which has area $\pi(1 + K_{\max})^2\lambda_1^{-2n}$. Since the interiors of the parallelograms are pairwise disjoint, the maximum number of intersecting elements per ball is bounded by $c_2 = \frac{\pi(1+K_{\max})^2}{K_{\min}}$, which completes the proof. □

Corollary 2.8. *The additive constants $\log c_1$ and $\log c_2$ vanish in the scaling limit, ensuring that the Minkowski dimension satisfies:*

$$\overline{\dim}_M Y = \limsup_{n\to\infty} \frac{\log \widetilde{N}(Y, n)}{n \log \lambda_1} \tag{4}$$

2.4 Topological Entropy and Dimension

We relate the topological entropy of Y to the symbolic space Σ_A.

Claim 2.9. *The semi-conjugacy $\phi : \Sigma_A \to \mathbb{T}^2$ is finite-to-one.*

Proof. By construction, ϕ is strictly injective except on the boundaries of the partition rectangles R_i. Because these boundaries consist of a finite union of lower-dimensional stable and unstable segments, any point has at most 4 pre-images. □

Corollary 2.10. *By standard ergodic principles for finite-to-one extensions, the entropy is preserved: $h_{top}(\phi^{-1}(Y)) = h_{top}(Y)$.*

Claim 2.11. *For the symbolic subsystem $\Sigma_Y = \phi^{-1}(Y)$, the topological entropy corresponds to the language growth rate:*

$$h_{top}(\Sigma_Y) = \lim_{k\to\infty} \frac{\log |L_k(\Sigma_Y)|}{k} \tag{5}$$

Claim 2.12. *The count $\widetilde{N}(Y, n)$ matches the number of admissible words of length $2n + 1$ in Σ_Y.*

Proof. Each unique element of $\mathcal{B}_{n,n}$ maps to a word $(a_{-n}, \ldots, a_n)$ of length $2n + 1$. Therefore we have: $\widetilde{N}(Y, n) = |L_{2n+1}(\Sigma_Y)|$. □

Combining these results, we establish the explicit calculation:

$$h_{\text{top}}(Y) = \lim_{n\to\infty} \frac{\log \widetilde{N}(Y, n)}{2n + 1}$$

Thus, $\dim_M Y$ exists and simplifies directly to:

$$\dim_M Y = \frac{2h_{\text{top}}(Y)}{\log \lambda_1} \tag{6}$$

2.5 Hausdorff Dimension Equivalence

Let $\widehat{\mathcal{B}}_{n,n}$ denote the collection $\mathcal{B}_{n,n}$ with the right and lower boundaries removed to ensure strict disjointness. Let μ be a B-invariant ergodic probability measure supported on Y. By the Shannon–McMillan–Breiman theorem, for μ-almost every x:

$$\lim_{n\to\infty} -\frac{1}{2n + 1} \log \mu(\widehat{\mathcal{B}}_{n,n}(x)) = h_\mu$$

By the mass distribution principle, it follows that $\dim_H Y \geq \frac{2}{\log \lambda_1} h_\mu$. Invoking the variational principle over all invariant measures, we obtain:

$$\dim_H Y \geq \frac{2}{\log \lambda_1} h_{\text{top}}(Y) = \dim_M Y$$

Since the reverse direction $\dim_H Y \leq \dim_M Y$ always holds, we conclude $\dim_H X = \dim_M X$ for all invariant subsystems.

3 Matrices with Equal Eigenvalues

Theorem 3.1. *Let $A \in M_2(\mathbb{Z})$ be a toral endomorphism with equal eigenvalues $\lambda_1 = \lambda_2 = \lambda > 1$. For any invariant subset $X \subseteq \mathbb{T}^2$, we have $\dim_H X = \dim_M X$.*

Proposition 3.2. *Let $A \in M_2(\mathbb{Z})$ have a repeated eigenvalue $\lambda > 1$ and a non-trivial Jordan block. There exist constants $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$ and all vectors $z \in \mathbb{R}^2$:*

$$C_1 \frac{\lambda^n}{n} \|z\| \leq \|A^n z\| \leq C_2 n \lambda^n \|z\| \tag{7}$$

Proof. Under the assumptions, there exists an invertible matrix P such that $A = PJP^{-1}$, where $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. The n-th matrix power is explicitly:

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} = \lambda^n \begin{pmatrix} 1 & n/\lambda \\ 0 & 1 \end{pmatrix}$$

Let $S_n = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ where $\alpha = n/\lambda$. The singular values $\sigma_1 \geq \sigma_2$ of S_n are the square roots of the eigenvalues of $M = S_n^T S_n = \begin{pmatrix} 1 & \alpha \\ \alpha & \alpha^2 + 1 \end{pmatrix}$. The characteristic equation $\det(M - \mu I) = 0$ is:

$$\mu^2 - (\alpha^2 + 2)\mu + 1 = 0$$

Evaluating via the quadratic formula yields the explicit matrix roots:

$$\mu_\pm = \frac{\alpha^2 + 2 \pm \alpha\sqrt{\alpha^2 + 4}}{2}$$

For the maximal root, we observe $\alpha^2 < \mu_+ < \alpha^2 + 2$. Thus, the largest singular value of J^n satisfies $\sigma_1(J^n) = \lambda^n \sqrt{\mu_+} \leq Cn\lambda^n$. For the minimal root, since $\mu_+ \mu_- = 1$, we have $\mu_- = 1/\mu_+$. Given $\mu_+ < (n/\lambda + 1)^2$ for large n, we obtain $\mu_- > (n/\lambda + 1)^{-2}$, hence $\sigma_2(J^n) \geq C'\lambda^n/n$. Applying standard matrix norm equivalence bounds to $A^n = PJ^nP^{-1}$ completes the proof. □

To handle the polynomial growth adjustments, we introduce a customized variant of Falconer's theorem [2]:

Proposition 3.3. *Let (F, d) be a compact metric space. Suppose there exist constants $a > 0$ and $0 < m \leq 1$ such that for every subset $N \subseteq F$ with sufficiently small diameter, there is a localized mapping $\phi_N : N \to F$ satisfying:*

$$a^m \cdot d(x, y) \leq \operatorname{diam}(N)^m \cdot d(\phi_N(x), \phi_N(y)) \quad \forall x, y \in N \tag{8}$$

Then the upper dimension satisfies $\overline{\dim}_M(F) \leq \frac{1}{m} \dim_H(F)$.

Proof. Assume $s > \frac{\dim_H F}{m}$, which implies $\mathcal{H}^{sm}(F) = 0 < a^{sm}$. Let $U_1, \ldots, U_k$ be an open cover of F with small diameters $|U_i| < \lambda_0$ and $|U_i|^m < a^m/2$ satisfying $\sum_{i=1}^k |U_i|^{sm} < a^{sm}$. By hypothesis, we have mappings $\varphi_i : U_i \to F$ stretching by at least $a^m |U_i|^{-m}$.

Let J_q be the set of q-term sequences chosen from $\{1, \ldots, k\}$. For each multi-index $(i_1, \ldots, i_q) \in J_q$, we define the nested sets:

$$U_{i_1,\ldots,i_q} = \varphi_{i_1}^{-1}\left(\varphi_{i_2}^{-1}\left(\ldots\left(\varphi_{i_q}^{-1}(F)\right)\right)\right)$$

Repeated application of the contraction properties yields the geometric bound:

$$|U_{i_1,\ldots,i_q}| \leq a^{-qm} |U_{i_1}|^m \ldots |U_{i_q}|^m |F| \tag{9}$$

Let $b = \min_{1\le i\le k} a^{-m}|U_i|^m$. For any resolution scale $\epsilon < |F|$, we terminate each infinite branch at the unique minimal index step q where:

$$b\epsilon < \left(a^{-1}|U_{i_1}|\right)^m \dots \left(a^{-1}|U_{i_q}|\right)^m |F| \le \epsilon \tag{10}$$

Let S denote the set of all such cut-off finite sequences. This configuration provides a finite covering $F \subset \bigcup_S U_{i_1,\dots,i_q}$. Summing the geometric steps yields:

$$(b\epsilon)^s \operatorname{card} S < \sum_S \left(a^{-1}|U_{i_1}|\right)^{sm} \dots \left(a^{-1}|U_{i_q}|\right)^{sm} |F|^{sm} < |F|^s \tag{11}$$

Hence, F can be covered by at most $|F|^s(b\epsilon)^{-s}$ sets of diameter ϵ, showing $\overline{\dim}_M F \le s$. Taking $s \to \frac{\dim_H F}{m}$ yields the conclusion. □

Proof of Theorem 3.1. If A is completely diagonalizable, the result follows from standard product space techniques. We therefore assume A possesses a non-trivial Jordan block. Fix any parameter $m \in (0,1)$. We check the validation requirements of Proposition 3.3 for this m setting $a = 1$. Let $N \subseteq X$ be an arbitrary tracking subset with sufficiently small diameter. We select the unique index $n \in \mathbb{N}$ tracking the scale:

$$\frac{1}{10C_2(n+1)\lambda^{n+1}} < \operatorname{diam}(N) \le \frac{1}{10C_2 n\lambda^n}$$

From the right inequality and Proposition 3.2, the global expansion is bounded:

$$\operatorname{diam}(A^n(N)) \le C_2 n\lambda^n \operatorname{diam}(N) \le \frac{1}{10} < \frac{1}{2}$$

This implies the metric tracking on $\mathbb{T}^2$ matches the standard Euclidean metric exactly. Leveraging the left tracking bound, we obtain $\operatorname{diam}(N)^{-m} < \left(10C_2(n+1)\lambda^{n+1}\right)^m$. Simultaneously, Proposition 3.2 ensures:

$$d(A^n x, A^n y) \ge C_1 \frac{\lambda^n}{n} d(x,y) \quad \forall x, y \in N$$

Since $\lambda > 1$ and $m < 1$, the exponential term λ^n dominates the polynomial step n and the scaled power $\lambda^{m(n+1)}$ for all large n. Thus, there exists a threshold index n_0 such that for all $n \ge n_0$:

$$C_1 \frac{\lambda^n}{n} \ge \left(10C_2(n+1)\lambda^{n+1}\right)^m > \operatorname{diam}(N)^{-m}$$

Setting $\delta = \frac{1}{10C_2 n_0 \lambda^{n_0}}$, any subset with $\operatorname{diam}(N) < \delta$ forces $n \ge n_0$. Setting $\phi_N = A^n$, we fulfill the conditions of Proposition 3.3, yielding $\overline{\dim}_M X \le \frac{1}{m}\dim_H X$. Taking the limit as $m \to 1^-$ concludes the proof. □

4 Hyperbolic Non-Invertible Matrices with Mixed Eigenvalues

We now investigate the non-invertible hyperbolic case characterized by eigenvalues satisfying $0 < \lambda_1 < 1 < \lambda_2$. We begin our local analysis using the specific matrix:

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} \tag{12}$$

and subsequently extend our findings to general configurations in $M_2(\mathbb{Z})$.

We note that Falconer's Theorem cannot be applied in the same straightforward manner as in the previous section. Furthermore, because A is not an Anosov diffeomorphism (as it lacks global invertibility), we cannot immediately construct a standard Markov partition to obtain a clean symbolic coding. To resolve this, we construct a custom Markov partition on the corresponding natural extension (the invertible lift space).

4.1 A Markov Partition for the Invertible Lift

We attempt to use slight variations of Bowen's methods to obtain a Markov partition in the inverse limit space of the system $(\mathbb{T}^2, A)$, defined as:

$$\hat{\mathbb{T}}^2 = \{\mathbf{x} = (x_0, x_1, x_2, \dots) : x_i \in \mathbb{T}^2 \text{ and } Ax_i = x_{i-1} \text{ for all } i \in \mathbb{N}\}$$

with the natural lift map $\bar{A} : \hat{\mathbb{T}}^2 \to \hat{\mathbb{T}}^2$, which is now invertible.

Let us clearly define the action of $\bar{A}$ and its inverse $\bar{A}^{-1}$ on this space. The map $\bar{A}$ applies A to all coordinates:

$$(\bar{A}\mathbf{x})_i = Ax_i = \begin{cases} Ax_0 & \text{for } i = 0 \\ x_{i-1} & \text{for } i \geq 1 \end{cases}$$

Thus, $\bar{A}\mathbf{x} = (Ax_0, x_0, x_1, x_2, \dots)$. Conversely, the inverse map $\bar{A}^{-1}$ simply shifts the sequence one index to the left:

$$(\bar{A}^{-1}\mathbf{x})_i = x_{i+1}$$

So $\bar{A}^{-1}\mathbf{x} = (x_1, x_2, x_3, \dots)$. We equip $\hat{\mathbb{T}}^2$ with the metric:

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=0}^{\infty} \frac{1}{2^i} d_{\mathbb{T}^2}(x_i, y_i)$$

Using this precise metric and the shift mappings, we can rigorously characterize the stable and unstable manifolds in the inverse limit space.

Claim 4.1 (Local Stable Manifold in $\hat{\mathbb{T}}^2$)**.** *For $\mathbf{x} \in \hat{\mathbb{T}}^2$ and small $\epsilon > 0$, the local stable manifold is given by:*

$$W^s_\epsilon(\mathbf{x}, \bar{A}) = \{\mathbf{y} \in \hat{\mathbb{T}}^2 : y_0 \in W^s_\epsilon(x_0, A) \text{ and } d(\mathbf{x}, \mathbf{y}) \leq \epsilon\}$$

Proof. By definition, we require $d(\bar{A}^n\mathbf{x}, \bar{A}^n\mathbf{y}) \leq \epsilon$ for all $n \geq 0$. As $n \to \infty$, the i-th coordinate of $\bar{A}^n\mathbf{x}$ is $A^{n-i}x_0$ for $i \leq n$. For the total sum to remain bounded by ϵ, the forward iterates $A^k x_0$ and $A^k y_0$ must track each other, which strictly requires $y_0 \in W^s_\epsilon(x_0, A)$. The condition $d(\mathbf{x}, \mathbf{y}) \leq \epsilon$ guarantees the initial metric bound for $n = 0$. Any valid sequence of preimages $\mathbf{y}$ satisfying these two conditions will naturally satisfy the bound for all $n > 0$ because the shift map moves the sequence indices to the right, heavily discounting them by the metric's 2^{-i} factor. □

Claim 4.2 (Local Unstable Manifold in $\hat{\mathbb{T}}^2$)**.** *For $\mathbf{x} \in \hat{\mathbb{T}}^2$ and small $\epsilon > 0$, the local unstable manifold requires strict alignment along the unstable leaves of A for every coordinate:*

$$W^u_\epsilon(\mathbf{x}, \bar{A}) = \{\mathbf{y} \in \hat{\mathbb{T}}^2 : y_i \in W^u_\epsilon(x_i, A) \text{ for all } i \geq 0\}$$

Proof. We require $d(\bar{A}^{-n}\mathbf{x}, \bar{A}^{-n}\mathbf{y}) \leq \epsilon$ for all $n \geq 0$. Since $(\bar{A}^{-n}\mathbf{x})_i = x_{i+n}$, we must have $d_{\mathbb{T}^2}(x_m, y_m) \leq \epsilon$ uniformly for all $m \geq 0$. Moving forward in the sequence indices equates to taking preimages, which applies the local branches of A^{-1}. Because A^{-1} expands the stable subspace, $y_m - x_m$ must have zero stable component to remain bounded. Thus, y_i must lie entirely in $W^u_\epsilon(x_i, A)$ for all $i \geq 0$. Because A^{-1} contracts the unstable leaves, the distances $d_{\mathbb{T}^2}(x_i, y_i)$ strictly decay to 0 as $i \to \infty$.

□

Claim 4.3 (Canonical Coordinates in $\hat{\mathbb{T}}^2$)**.** *For $\mathbf{x}, \mathbf{y} \in \hat{\mathbb{T}}^2$ sufficiently close, the canonical intersection $[\mathbf{x}, \mathbf{y}] = W^s_\epsilon(\mathbf{x}, \bar{A}) \cap W^u_\epsilon(\mathbf{y}, \bar{A})$ consists of a uniquely defined sequence $\mathbf{z} \in \hat{\mathbb{T}}^2$ generated strictly by the unstable branches:*

$$z_0 = [x_0, y_0]_A = W^s_\epsilon(x_0, A) \cap W^u_\epsilon(y_0, A)$$

$$z_i = (A|_{W^u(y_i, A)})^{-1}(z_{i-1}) \quad \text{for all } i \geq 1$$

Proof. Since $\mathbf{z} \in W^s_\epsilon(\mathbf{x}, \bar{A})$, Claim 1 requires $z_0 \in W^s_\epsilon(x_0, A)$. Since $\mathbf{z} \in W^u_\epsilon(\mathbf{y}, \bar{A})$, Claim 2 requires $z_i \in W^u_\epsilon(y_i, A)$ for all $i \geq 0$. For the 0-th coordinate, this forces $z_0 \in W^s_\epsilon(x_0, A) \cap W^u_\epsilon(y_0, A)$, uniquely defining z_0 on the torus.

For $i \geq 1$, the inverse limit structure requires $Az_i = z_{i-1}$. Because A restricted to its unstable leaves is a bijective expansion, its inverse is a strict contraction. Therefore, z_i is uniquely determined by induction as the sole valid preimage of z_{i-1} residing inside $W^u(y_i, A)$. Finally, because $d(y_i, z_i) \to 0$ exponentially, the sum $d(\mathbf{x}, \mathbf{z})$ satisfies the ϵ bound required to safely place the entirely unique sequence $\mathbf{z}$ inside $W^s_\epsilon(\mathbf{x}, \bar{A})$.

□

Definition 4.4. A sequence $\{\mathbf{x}^{(i)}\}_{i=-\infty}^{\infty}$ of points in $\hat{\mathbb{T}}^2$ is an **α-pseudo-orbit** if:

$$d(\bar{A}\mathbf{x}^{(i)}, \mathbf{x}^{(i+1)}) < \alpha \quad \text{for all } i \in \mathbb{Z}$$

Definition 4.5. A point $\mathbf{x} \in \hat{\mathbb{T}}^2$ **β-shadows** a pseudo-orbit $\{\mathbf{x}^{(i)}\}$ if:

$$d(\bar{A}^i\mathbf{x}, \mathbf{x}^{(i)}) \leq \beta \quad \text{for all } i$$

Proposition 4.6. *For every $\beta > 0$, there exists $\alpha > 0$ such that every bi-infinite α-pseudo-orbit $\{\mathbf{x}^{(i)}\}_{i=-\infty}^{\infty}$ in $\hat{\mathbb{T}}^2$ is β-shadowed by some point $\mathbf{x} \in \hat{\mathbb{T}}^2$.*

Proof. Let $\varepsilon > 0$ such that $\beta' := \frac{\varepsilon\lambda_1}{1-\lambda_1} + \varepsilon + \frac{\varepsilon}{2} < \beta$.

Let $\delta \leq \frac{\varepsilon}{2}$, so $W^s_\varepsilon(\mathbf{y}) \cap W^u_\varepsilon(\mathbf{z}) \neq \emptyset$ for $d(\mathbf{y}, \mathbf{z}) \leq \delta$.

Pick $M \in \mathbb{N}$ such that $\lambda_1^M \varepsilon < \delta/2$ and choose $\alpha > 0$ such that any α-pseudo-orbit $\{\mathbf{y}^{(i)}\}_{i=0}^{M}$ satisfies $d(\bar{A}^j\mathbf{y}^{(0)}, \mathbf{y}^{(j)}) < \delta/2$ for all $j \in [0, M]$.

Consider a bi-infinite α-pseudo-orbit $\{\mathbf{x}^{(i)}\}_{i=-\infty}^{\infty}$. For each $r \in \mathbb{N}$, consider its finite truncation on the symmetric interval $[-rM, rM]$. Define $\mathbf{z}^{(kM)}$ recursively for $k \in [-r, r-1]$ by starting at the left endpoint $\mathbf{z}^{(-rM)} = \mathbf{x}^{(-rM)}$ and setting:

$$\mathbf{z}^{((k+1)M)} = W^u_\varepsilon(\bar{A}^M \mathbf{z}^{(kM)}) \cap W^s_\varepsilon(\mathbf{x}^{((k+1)M)}) \in \hat{\mathbb{T}}^2$$

This is well-defined since $d(\bar{A}^M \mathbf{z}^{(kM)}, \mathbf{x}^{((k+1)M)}) \leq d(\bar{A}^M \mathbf{z}^{(kM)}, \bar{A}^M \mathbf{x}^{(kM)}) + d(\bar{A}^M \mathbf{x}^{(kM)}, \mathbf{x}^{((k+1)M)}) < \lambda_1^M \varepsilon + \delta/2 < \delta$.

To find the shadowing point at index 0, we set $\mathbf{w}_r = \bar{A}^{-rM} \mathbf{z}^{(rM)}$. For any $i \in [sM, (s+1)M)$ with $-r \leq s < r$, we split the distance $d(\bar{A}^i \mathbf{w}_r, \mathbf{x}^{(i)})$ into three components via the triangle inequality:

$$\begin{aligned} \mathrm{d}({}^i\mathbf{w}_r, \mathbf{x}^{(i)}) &\leq d(\bar{A}^i \mathbf{w}_r, \bar{A}^{i-sM} \mathbf{z}^{(sM)}) \\ &+ \mathrm{d}({}^{i-sM}\mathbf{z}^{(sM)}, \bar{A}^{i-sM} \mathbf{x}^{(sM)}) \\ &+ \mathrm{d}({}^{i-sM}\mathbf{x}^{(sM)}, \mathbf{x}^{(i)}) \end{aligned}$$

We bound each term separately. For the first term, we write $\bar{A}^i \mathbf{w}_r = \bar{A}^{i-rM} \mathbf{z}^{(rM)}$ and expand it into a telescoping sum over intervals of length M. Because $\mathbf{z}^{(tM)} \in W^u_\varepsilon(\bar{A}^M \mathbf{z}^{((t-1)M)})$, applying $\bar{A}^{-1}$ exactly $tM - i$ times contracts distances along the unstable manifold by λ_1:

$$d(\bar{A}^i \mathbf{w}_r, \bar{A}^{i-sM} \mathbf{z}^{(sM)}) \leq \sum_{t=s+1}^{r} d\left(\bar{A}^{i-tM} \mathbf{z}^{(tM)}, \bar{A}^{i-tM}(\bar{A}^M \mathbf{z}^{((t-1)M)})\right) \leq \sum_{t=s+1}^{r} \varepsilon \lambda_1^{tM-i}$$

For the second term, since $\mathbf{z}^{(sM)} \in W^s_\varepsilon(\mathbf{x}^{(sM)})$, forward iteration by $i - sM \geq 0$ steps contracts the distance along the stable manifold, giving a strict upper bound of ε.

For the third term, the forward interval length is $i - sM \in [0, M)$. By our choice of α, this segment is tracked within $\delta/2$.

Combining these bounds yields that for any $i \in [-rM, rM]$:

$$d(\bar{A}^i \mathbf{w}_r, \mathbf{x}^{(i)}) \leq \sum_{t=s+1}^{r} \varepsilon \lambda_1^{tM-i} + \varepsilon + \frac{\delta}{2} \leq \frac{\varepsilon \lambda_1}{1 - \lambda_1} + \varepsilon + \frac{\delta}{2} = \beta' < \beta$$

Thus, $\mathbf{w}_r$ is a point that strictly β'-shadows the finite pseudo-orbit segment from index $-rM$ to rM.

To extend this to the bi-infinite pseudo-orbit, we observe that the inverse limit space $\hat{\mathbb{T}}^2$ is compact (by Tychonoff's Theorem). Therefore, the sequence of index-zero shadowing points $\{\mathbf{w}_r\}_{r=1}^{\infty}$ admits a convergent subsequence $\mathbf{w}_{r_k} \to \mathbf{x} \in \hat{\mathbb{T}}^2$ as $k \to \infty$.

Fix any index $i \in \mathbb{Z}$. For all k large enough such that $-r_k M \leq i \leq r_k M$, we have $d(\bar{A}^i \mathbf{w}_{r_k}, \mathbf{x}^{(i)}) \leq \beta'$. By the continuity of the metric d and the map $\bar{A}$, taking the limit as $k \to \infty$ gives:

$$d(\bar{A}^i \mathbf{x}, \mathbf{x}^{(i)}) = \lim_{k \to \infty} d(\bar{A}^i \mathbf{w}_{r_k}, \mathbf{x}^{(i)}) \leq \beta' < \beta$$

Since $i \in \mathbb{Z}$ was arbitrary, the limit point $\mathbf{x}$ globally β-shadows the entire bi-infinite pseudo-orbit.

□

4.2 Markov Partitions

Lemma 4.7 (Expansiveness of the Lift)**.** *The invertible lift map $\bar{A} : \hat{\mathbb{T}}^2 \to \hat{\mathbb{T}}^2$ is expansive.*

Proof. Because A is an expanding/hyperbolic endomorphism on $\mathbb{T}^2$, there exists an expansiveness constant for the base system. In the inverse limit space $\hat{\mathbb{T}}^2$, the metric $d(\mathbf{x}, \mathbf{y}) = \sum_{i=0}^{\infty} \frac{1}{2^i} d_{\mathbb{T}^2}(x_i, y_i)$ dictates that if $d(\bar{A}^n \mathbf{x}, \bar{A}^n \mathbf{y})$ is bounded for all $n \in \mathbb{Z}$, the 0-th coordinates must track each other indefinitely under both forward iteration (via A) and backward tracking (via the specific sequence of preimages). Transversality of the local stable and unstable leaves on $\mathbb{T}^2$ then forces $x_0 = y_0$, and consequently $\mathbf{x} = \mathbf{y}$ for the entire sequence. □

Theorem 4.8. *$\hat{\mathbb{T}}^2$ has Markov partitions $\mathcal{R}$ of arbitrarily small diameter.*

Proof. Let $\beta = 0.001$, $\alpha > 0$ as in the shadowing proposition. Choose $\gamma < \alpha/2$ so that

$$d(\bar{A}\mathbf{x}, \bar{A}\mathbf{y}) < \alpha/2 \quad \text{when} \quad d(\mathbf{x}, \mathbf{y}) < \gamma$$

Let $P = \{\mathbf{p}_1, \ldots, \mathbf{p}_r\}$ be a γ-dense subset of $\hat{\mathbb{T}}^2$ and

$$\Sigma(P) = \left\{ \mathbf{q} \in \prod_{j=-\infty}^{\infty} P : d\left(\bar{A}\mathbf{q}^{(j)}, \mathbf{q}^{(j+1)}\right) < \alpha \text{ for all } j \right\}.$$

For each $\mathbf{q} \in \Sigma(P)$ there is a unique $\theta(\mathbf{q}) \in \hat{\mathbb{T}}^2$ which β-shadows $\mathbf{q}$; for each $\mathbf{x} \in \hat{\mathbb{T}}^2$ there is a sequence $\mathbf{q}$ with $\mathbf{x} = \theta(\mathbf{q})$.

For $\mathbf{q}, \mathbf{q}' \in \Sigma(P)$ with $\mathbf{q}^{(0)} = \mathbf{p}_s = (\mathbf{q}')^{(0)}$ we define $\mathbf{q}^* = [\mathbf{q}, \mathbf{q}'] \in \Sigma(P)$ by

$$(\mathbf{q}^*)^{(j)} = \begin{cases} \mathbf{q}^{(j)} & \text{for } j \geq 0 \\ (\mathbf{q}')^{(j)} & \text{for } j \leq 0 \end{cases}$$

Then $d\left(\bar{A}^j \theta\left(\mathbf{q}^*\right), \bar{A}^j \theta(\mathbf{q})\right) \leq 2\beta$ for $j \geq 0$ and $d\left(\bar{A}^j \theta\left(\mathbf{q}^*\right), \bar{A}^j \theta(\mathbf{q}')\right) \leq 2\beta$ for $j \leq 0$. So $\theta\left(\mathbf{q}^*\right) \in W^s_{2\beta}(\theta(\mathbf{q})) \cap W^u_{2\beta}\left(\theta\left(\mathbf{q}'\right)\right)$, i.e.,

$$\theta\left[\mathbf{q}, \mathbf{q}'\right] = \left[\theta(\mathbf{q}), \theta\left(\mathbf{q}'\right)\right]$$

We now see that $T_s = \left\{\theta(\mathbf{q}) : \mathbf{q} \in \Sigma(P), \mathbf{q}^{(0)} = \mathbf{p}_s\right\}$ is a rectangle. For $\mathbf{x}, \mathbf{y} \in T_s$ we write $\mathbf{x} = \theta(\mathbf{q}), \mathbf{y} = \theta\left(\mathbf{q}'\right)$ with $\mathbf{q}^{(0)} = \mathbf{p}_s = (\mathbf{q}')^{(0)}$. Then

$$[\mathbf{x}, \mathbf{y}] = \theta\left[\mathbf{q}, \mathbf{q}'\right] \in T_s$$

Suppose $\mathbf{x} = \theta(\mathbf{q})$ with $\mathbf{q}^{(0)} = \mathbf{p}_s$ and $\mathbf{q}^{(1)} = \mathbf{p}_t$. Consider $\mathbf{y} \in W^s\left(\mathbf{x}, T_s\right), \mathbf{y} = \theta\left(\mathbf{q}'\right)$, $\mathbf{q}^{(0)} = \mathbf{p}_s$. Then

$$\begin{aligned} \mathbf{y} = [\mathbf{x}, \mathbf{y}] = \theta\left[\mathbf{q}, \mathbf{q}'\right] \quad &\text{and} \\ \bar{A}\mathbf{y} = \theta\left(\sigma\left[\mathbf{q}, \mathbf{q}'\right]\right) \in T_t \end{aligned}$$

as $\sigma\left[\mathbf{q}, \mathbf{q}'\right]$ has $(\mathbf{q}')^{(1)} = \mathbf{p}_t$ in its zeroth position. Since $\bar{A}\mathbf{y} \in W^s_\epsilon(\bar{A}\mathbf{x})$ ($\operatorname{diam}\left(T_s\right) \leq 2\beta$ is small compared to ϵ), $\bar{A}\mathbf{y} \in W^s_\epsilon\left(\bar{A}\mathbf{x}, T_t\right)$. We have proved:

[(i)]

1. $\bar{A}W^s\left(\mathbf{x}, T_s\right) \subset W^s\left(\bar{A}\mathbf{x}, T_t\right)$.

A similar proof shows $\bar{A}^{-1}W^u\left(\bar{A}\mathbf{x}, T_t\right) \subset W^u\left(\mathbf{x}, T_s\right)$, i.e.,

[(i)]

2. $\bar{A}W^u\left(\mathbf{x}, T_s\right) \supset W^u\left(\bar{A}\mathbf{x}, T_t\right)$.

Each T_s is closed; this follows from the following lemma. □

Lemma 4.9. *$\theta : \Sigma(P) \to \hat{\mathbb{T}}^2$ is continuous.*

Proof. Otherwise there is a $\gamma > 0$ so that for every N one can find $\mathbf{q}_N, \mathbf{q}'_N \in \Sigma(P)$ with $\mathbf{q}_N^{(j)} = (\mathbf{q}'_N)^{(j)}$ for all $j \in [-N, N]$ but $d\left(\theta\left(\mathbf{q}_N\right), \theta\left(\mathbf{q}'_N\right)\right) \geq \gamma$. If $\mathbf{x}_N = \theta\left(\mathbf{q}_N\right), \mathbf{y}_N = \theta\left(\mathbf{q}'_N\right)$ one has

$$d\left(\bar{A}^j \mathbf{x}_N, \bar{A}^j \mathbf{y}_N\right) \leq 2\beta \quad \forall j \in [-N, N]$$

Taking subsequences we may assume $\mathbf{x}_N \to \mathbf{x}$ and $\mathbf{y}_N \to \mathbf{y}$ as $N \to \infty$. Then $d\left(\bar{A}^j \mathbf{x}, \bar{A}^j \mathbf{y}\right) \leq 2\beta$ for all j and $d(\mathbf{x}, \mathbf{y}) \geq \gamma$; this contradicts the expansiveness of $\bar{A}$ on $\hat{\mathbb{T}}^2$. □

Now $\mathcal{T} = \{T_1, \ldots, T_r\}$ is a covering by rectangles and (i) and (ii) above act as the Markov condition (b). However, the T_j's are likely to overlap and not be proper. For each $\mathbf{x} \in \hat{\mathbb{T}}^2$ let $\mathcal{T}(\mathbf{x}) = \{T_j \in \mathcal{T} : \mathbf{x} \in T_j\}$ and $\mathcal{T}^*(\mathbf{x}) = \{T_k \in \mathcal{T} : T_k \cap T_j \neq \emptyset \text{ for some } T_j \in \mathcal{T}(\mathbf{x})\}$.

As $\mathcal{T}$ is a closed cover of $\hat{\mathbb{T}}^2$, $Z = \hat{\mathbb{T}}^2 \setminus \bigcup_j \partial T_j$ is an open dense subset of $\hat{\mathbb{T}}^2$. In fact, one can show that

$$Z^* = \left\{\mathbf{x} \in \hat{\mathbb{T}}^2 : W^s_\epsilon(\mathbf{x}) \cap \partial^s T_k = \emptyset \text{ and } W^u_\epsilon(\mathbf{x}) \cap \partial^u T_k = \emptyset \text{ for all } T_k \in \mathcal{T}^*(\mathbf{x})\right\}$$

is open and dense in $\hat{\mathbb{T}}^2$.

For $T_j \cap T_k \neq \emptyset$, let

$$\begin{aligned}
T^1_{j,k} &= \{\mathbf{x} \in T_j : W^u(\mathbf{x}, T_j) \cap T_k \neq \emptyset, W^s(\mathbf{x}, T_j) \cap T_k \neq \emptyset\} = T_j \cap T_k \\
T^2_{j,k} &= \{\mathbf{x} \in T_j : W^u(\mathbf{x}, T_j) \cap T_k \neq \emptyset, W^s(\mathbf{x}, T_j) \cap T_k = \emptyset\} \\
T^3_{j,k} &= \{\mathbf{x} \in T_j : W^u(\mathbf{x}, T_j) \cap T_k = \emptyset, W^s(\mathbf{x}, T_j) \cap T_k \neq \emptyset\} \\
T^4_{j,k} &= \{\mathbf{x} \in T_j : W^u(\mathbf{x}, T_j) \cap T_k = \emptyset, W^s(\mathbf{x}, T_j) \cap T_k = \emptyset\}
\end{aligned}$$

If $\mathbf{x}, \mathbf{y} \in T_j$, then $W^s([\mathbf{x}, \mathbf{y}], T_j) = W^s(\mathbf{x}, T_j)$ and $W^u([\mathbf{x}, \mathbf{y}], T_j) = W^u(\mathbf{y}, T_j)$; this implies $T^n_{j,k}$ is a rectangle open in $\hat{\mathbb{T}}^2$ and each $\mathbf{x} \in T_j \cap Z^*$ lies in $\operatorname{int}\left(T^n_{j,k}\right)$ for some n. For $\mathbf{x} \in Z^*$ define

$$R(\mathbf{x}) = \bigcap \left\{\operatorname{int}\left(T^n_{j,k}\right) : \mathbf{x} \in T_j, T_k \cap T_j \neq \emptyset \text{ and } \mathbf{x} \in T^n_{j,k}\right\}.$$

Now $R(\mathbf{x})$ is an open rectangle ($\mathbf{x} \in Z^*$). Suppose $\mathbf{y} \in R(\mathbf{x}) \cap Z^*$. Since $R(\mathbf{x}) \subset \mathcal{T}(\mathbf{x})$ and $R(\mathbf{x}) \cap T_j = \emptyset$ for $T_j \notin \mathcal{T}(\mathbf{x})$, one gets $\mathcal{T}(\mathbf{y}) = \mathcal{T}(\mathbf{x})$. For $T_j \in \mathcal{T}(\mathbf{x}) = \mathcal{T}(\mathbf{y})$ and $T_k \cap T_j \neq \emptyset$, $\mathbf{y}$ lies in the same $T^n_{j,k}$ as $\mathbf{x}$ does since $T^n_{j,k} \supset R(\mathbf{x})$; hence $R(\mathbf{y}) = R(\mathbf{x})$. If $R(\mathbf{x}) \cap R\left(\mathbf{x}'\right) \neq \emptyset$ ($\mathbf{x}, \mathbf{x}' \in Z^*$), there is a $\mathbf{y} \in R(\mathbf{x}) \cap R\left(\mathbf{x}'\right) \cap Z^*$; then $R(\mathbf{x}) = R(\mathbf{y}) = R\left(\mathbf{x}'\right)$. As there are only finitely many $T^n_{j,k}$'s there are only finitely many distinct $R(\mathbf{x})$'s. Let

$$\mathcal{R} = \left\{\overline{R(\mathbf{x})} : \mathbf{x} \in Z^*\right\} = \{R_1, \ldots, R_m\}$$

For $\mathbf{x}' \in Z^*$, $R\left(\mathbf{x}'\right) = R(\mathbf{x})$ or $R\left(\mathbf{x}'\right) \cap R(\mathbf{x}) = \emptyset$; hence $(\overline{R(\mathbf{x})} \setminus R(\mathbf{x})) \cap Z^* = \emptyset$. As Z^* is dense in $\hat{\mathbb{T}}^2$, $\overline{R(\mathbf{x})} \setminus R(\mathbf{x})$ has no interior (in $\hat{\mathbb{T}}^2$) and $R(\mathbf{x}) = \operatorname{int}(\overline{R(\mathbf{x})})$. For $R(\mathbf{x}) \neq R\left(\mathbf{x}'\right)$

$$\operatorname{int}(\overline{R(\mathbf{x})}) \cap \operatorname{int}\left(\overline{R\left(\mathbf{x}'\right)}\right) = R(\mathbf{x}) \cap R\left(\mathbf{x}'\right) = \emptyset$$

To show that $\mathcal{R}$ is Markov we are left to verify condition (b). Suppose $\mathbf{x}, \mathbf{y} \in Z^* \cap \bar{A}^{-1} Z^*$, $R(\mathbf{x}) = R(\mathbf{y})$ and $\mathbf{y} \in W^s_\epsilon(\mathbf{x})$. We will show $R(\bar{A}\mathbf{x}) = R(\bar{A}\mathbf{y})$. First $\mathcal{T}(\bar{A}\mathbf{x}) = \mathcal{T}(\bar{A}\mathbf{y})$. Otherwise assume $\bar{A}\mathbf{x} \in T_j$, $\bar{A}\mathbf{y} \notin T_j$. Let $\bar{A}\mathbf{x} = \theta(\sigma\mathbf{q})$ with $\mathbf{q}^{(1)} = \mathbf{p}_j$ and $\mathbf{q}^{(0)} = \mathbf{p}_s$. Then $\mathbf{x} = \theta(\mathbf{q}) \in T_s$ and by inclusion (i) above

$$\bar{A}\mathbf{y} \in \bar{A} W^s(\mathbf{x}, T_s) \subset W^s\left(\bar{A}\mathbf{x}, T_j\right)$$

contradicting $\bar{A}\mathbf{y} \notin T_j$. Now let $\bar{A}\mathbf{x}, \bar{A}\mathbf{y} \in T_j$ and $T_k \cap T_j \neq \emptyset$. We want to show that $\bar{A}\mathbf{x}, \bar{A}\mathbf{y}$ belong to the same $T^n_{j,k}$. As $\bar{A}\mathbf{y} \in W^s_\epsilon(\bar{A}\mathbf{x})$ we have $W^s\left(\bar{A}\mathbf{y}, T_j\right) = W^s\left(\bar{A}\mathbf{x}, T_j\right)$. We will derive a contradiction from

$$W^u\left(\bar{A}\mathbf{y}, T_j\right) \cap T_k = \emptyset, \quad \bar{A}\mathbf{z} \in W^u\left(\bar{A}\mathbf{x}, T_j\right) \cap T_k$$

Recall that $\bar{A}\mathbf{x} = \theta(\sigma\mathbf{q})$, $\mathbf{q}^{(1)} = \mathbf{p}_j$, $\mathbf{q}^{(0)} = \mathbf{p}_s$. Then by inclusion (ii)

$$\bar{A}\mathbf{z} \in W^u\left(\bar{A}\mathbf{x}, T_j\right) \subset \bar{A}W^u\left(\mathbf{x}, T_s\right) \quad \text{or} \quad \mathbf{z} \in W^u\left(\mathbf{x}, T_s\right)$$

Let $\bar{A}\mathbf{z} = \theta\left(\sigma\mathbf{q}'\right)$; $\left(\mathbf{q}'\right)^{(1)} = \mathbf{p}_k$ and $\mathbf{q}^{(0)} = \mathbf{p}_t$. Then $\mathbf{z} \in T_t$ and $\bar{A}W^s\left(\mathbf{z}, T_t\right) \subset W^s\left(\bar{A}\mathbf{z}, T_k\right)$. Now $T_s \in \mathcal{T}(\mathbf{x}) = \mathcal{T}(\mathbf{y})$ and $\mathbf{z} \in T_t \cap T_s \neq \emptyset$. Now $\mathbf{z} \in W^u\left(\mathbf{x}, T_s\right) \cap T_t$ and so there is some $\mathbf{z}' \in W^u\left(\mathbf{y}, T_s\right) \cap T_t$ as $\mathbf{x}, \mathbf{y}$ are in the same $T^n_{s,t}$. Then

$$\mathbf{z}'' = [\mathbf{z}, \mathbf{y}] = \left[\mathbf{z}, \mathbf{z}'\right] \in W^s\left(\mathbf{z}, T_t\right) \cap W^u\left(\mathbf{y}, T_s\right)$$

and $\bar{A}\mathbf{z}'' = [\bar{A}\mathbf{z}, \bar{A}\mathbf{y}] \in W^s\left(\bar{A}\mathbf{z}, T_k\right) \cap W^u\left(\bar{A}\mathbf{y}, T_j\right)$ (using $\bar{A}\mathbf{z}, \bar{A}\mathbf{y} \in T_j$ a rectangle), a contradiction. So $R(\bar{A}\mathbf{x}) = R(\bar{A}\mathbf{y})$.

For small $\delta > 0$ the sets

$$Y_1 = \bigcup \left\{ W^s_\delta(\mathbf{z}) : \mathbf{z} \in \bigcup_j \partial^s T_j \right\} \quad \text{and} \quad Y_2 = \bigcup \left\{ W^u_\delta(\mathbf{z}) : \mathbf{z} \in \bigcup_j \partial^u T_j \right\}$$

are closed and nowhere dense. Now $Z^* \supset \hat{\mathbb{T}}^2 \setminus (Y_1 \cup Y_2)$ is open and dense. Furthermore if $\mathbf{x} \notin (Y_1 \cup Y_2) \cap \bar{A}^{-1}(Y_1 \cup Y_2)$ then $\mathbf{x} \in Z^* \cap \bar{A}^{-1}Z^*$ and the set of $\mathbf{y} \in W^s(\mathbf{x}, R(\mathbf{x}))$ with $\mathbf{y} \in Z^* \cap \bar{A}^{-1}Z^*$ is open and dense in $W^s(\mathbf{x}, \overline{R(\mathbf{x})})$ (as a subset of $W^s_\epsilon(\mathbf{x}) \cap \hat{\mathbb{T}}^2$). By the previous paragraph $R(\bar{A}\mathbf{y}) = R(\bar{A}\mathbf{x})$ for such $\mathbf{y}$; by continuity

$$\bar{A}W^s(\mathbf{x}, \overline{R(\mathbf{x})}) \subset \overline{R(\bar{A}\mathbf{x})}$$

As $\bar{A}W^s(\mathbf{x}, \overline{R(\mathbf{x})}) \subset W^s_\epsilon(\bar{A}\mathbf{x})$, $\bar{A}W^s(\mathbf{x}, \overline{R(\mathbf{x})}) \subset W^s(\bar{A}\mathbf{x}, \overline{R(\bar{A}\mathbf{x})})$. If $\operatorname{int}(R_i) \cap \bar{A}^{-1} \operatorname{int}(R_j) \neq \emptyset$, then this open subset of $\hat{\mathbb{T}}^2$ contains some $\mathbf{x}$ satisfying the above conditions, $R_i = \overline{R(\mathbf{x})}$ and $R_j = \overline{R(\bar{A}\mathbf{x})}$. For any $\mathbf{x}' \in R_i \cap \bar{A}^{-1}R_j$ one has $W^s(\mathbf{x}', R_i) = \{[\mathbf{x}', \mathbf{y}] : \mathbf{y} \in W^s(\mathbf{x}, R_i)\}$ and

$$\begin{aligned} \bar{A}W^s\left(\mathbf{x}', R_i\right) &= \left\{\left[\bar{A}\mathbf{x}', \bar{A}\mathbf{y}\right] : \mathbf{y} \in W^s\left(\mathbf{x}, R_i\right)\right\} \\ &\subset \left\{\left[\bar{A}\mathbf{x}', \mathbf{z}\right] : \mathbf{z} \in W^s\left(\bar{A}\mathbf{x}, R_j\right)\right\} \\ &\subset W^s\left(\bar{A}\mathbf{x}', R_j\right) \end{aligned}$$

This completes the proof of half of the Markov conditions (b). The other half is proved similarly and the proof is omitted. Alternatively one could apply the above to $\bar{A}^{-1}$, noting that $W^u_{\bar{A}} = W^s_{\bar{A}^{-1}}$.

4.3 Symbolic Dynamics

Throughout this section $\mathcal{R} = \{R_1, \ldots, R_m\}$ will denote a Markov partition of the basic set $\hat{\mathbb{T}}^2$. One defines the transition matrix $B = B(\mathcal{R})$ by

$$B_{ij} = \begin{cases} 1 & \text{if } \operatorname{int}(R_i) \cap \bar{A}^{-1} \operatorname{int}(R_j) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.10. *Suppose* $\mathbf{x} \in R_i, \bar{A}\mathbf{x} \in R_j, B_{ij} = 1$. *Then* $\bar{A}W^s(\mathbf{x}, R_i) \subset W^s\left(\bar{A}\mathbf{x}, R_j\right)$ *and* $\bar{A}W^u(\mathbf{x}, R_i) \supset W^u\left(\bar{A}\mathbf{x}, R_j\right)$.

Proof. This follows structurally from the final portion of the Markov partition theorem above. □

Definition 4.11. $\partial^s\mathcal{R} = \bigcup_j \partial^s R_j$ and $\partial^u\mathcal{R} = \bigcup_j \partial^u R_j$.

Proposition 4.12. *$\bar{A}\left(\partial^s\mathcal{R}\right) \subset \partial^s\mathcal{R}$ and $\bar{A}^{-1}\left(\partial^u\mathcal{R}\right) \subset \partial^u\mathcal{R}$.*

Proof. The set $\bigcup_j \left(\operatorname{int}(R_i) \cap \bar{A}^{-1}\operatorname{int}(R_j)\right)$ is dense in R_i. For any $\mathbf{x} \in R_i$ one can therefore find some j and $\mathbf{x}_n \in \operatorname{int}(R_i) \cap \bar{A}^{-1}\operatorname{int}(R_j)$ with $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$. Then $B_{ij} = 1$, $\mathbf{x} \in R_i$, and $\bar{A}\mathbf{x} \in R_j$. Hence $\bar{A}W^u(\mathbf{x}, R_i) \supset W^u\left(\bar{A}\mathbf{x}, R_j\right)$. If $\bar{A}\mathbf{x} \notin \partial^s\mathcal{R}$, then $W^u\left(\bar{A}\mathbf{x}, R_j\right)$ is a neighborhood of $\bar{A}\mathbf{x}$ in $W^s_\epsilon(\bar{A}\mathbf{x}) \cap \hat{\mathbb{T}}^2$ and so $W^u(\mathbf{x}, R_i)$ is a neighborhood of $\mathbf{x}$ in $W^s_\epsilon(\mathbf{x}) \cap \hat{\mathbb{T}}^2$—that is $\mathbf{x} \notin \partial^s R_i$. We have shown $\bar{A}\left(\partial^s\mathcal{R}\right) \subset \partial^s\mathcal{R}$. One gets $\bar{A}^{-1}\left(\partial^u\mathcal{R}\right) \subset \partial^u\mathcal{R}$ by a similar argument or by applying the first argument to $\bar{A}^{-1}$ in place of $\bar{A}$. □

Lemma 4.13. *Let $D \subset W^s_\delta(\mathbf{x}) \cap \hat{\mathbb{T}}^2$ and $C \subset W^u_\delta(\mathbf{x}) \cap \hat{\mathbb{T}}^2$. Then the rectangle $[C, D]$ is proper iff $D = \overline{\operatorname{int}(D)}$ and $C = \overline{\operatorname{int}(C)}$ as subsets of $W^s_\delta(\mathbf{x}) \cap \hat{\mathbb{T}}^2$ and $W^u_\delta(\mathbf{x}) \cap \hat{\mathbb{T}}^2$ respectively.*

Definition 4.14. Let R, S be two rectangles. S will be called a u-subrectangle of R if

[(a)]

1. $S \neq \emptyset, S \subset R$, S is proper, and
2. $W^u(\mathbf{y}, S) = W^u(\mathbf{y}, R)$ for $\mathbf{y} \in S$.

Lemma 4.15. *Suppose S is a u-subrectangle of R_i and $B_{ij} = 1$. Then $\bar{A}(S) \cap R_j$ is a u-subrectangle of R_j.*

Proof. Pick $\mathbf{x} \in R_i \cap \bar{A}^{-1}R_j$ and set $D = W^s(\mathbf{x}, R_i) \cap S$. Because S is a u-subrectangle (condition (b)) one has

$$S = \bigcup_{\mathbf{y}\in D} W^u(\mathbf{y}, R_i) = \left[W^u(\mathbf{x}, R_i), D\right].$$

As S is proper and nonempty, $D \neq \emptyset$ and $D = \overline{\operatorname{int}(D)}$. Now

$$\bar{A}(S) \cap R_j = \bigcup_{\mathbf{y}\in D} \left(\bar{A}W^u(\mathbf{y}, R_i) \cap R_j\right)$$

By the earlier lemma, $\bar{A}(\mathbf{y}) \in R_j$ and $\bar{A}W^u(\mathbf{y}, R_i) \cap R_j = W^u\left(\bar{A}\mathbf{y}, R_j\right)$. So $\bar{A}(S) \cap R_j = \bigcup_{\mathbf{y}'\in\bar{A}(D)} W^u(\mathbf{y}', R_j) = \left[W^u\left(\bar{A}\mathbf{x}, R_j\right), \bar{A}(D)\right]$.

Since $R_j = \left[W^u\left(\bar{A}\mathbf{x}, R_j\right), W^s\left(\bar{A}\mathbf{x}, R_j\right)\right]$ is proper, one has $W^u\left(\bar{A}\mathbf{x}, R_j\right)$ proper. As $\bar{A}$ maps $W^s_\epsilon(\mathbf{x}) \cap \hat{\mathbb{T}}^2$ homeomorphically onto a neighborhood in $W^s_\epsilon(\bar{A}\mathbf{x}) \cap \hat{\mathbb{T}}^2$, $\bar{A}(D) = \overline{\operatorname{int}(\bar{A}(D))}$ and so $\bar{A}(S) \cap R_j$ is proper. $\bar{A}(S) \cap R_j \neq \emptyset$ as $\bar{A}(D) \neq \emptyset$; if $\mathbf{y}'' \in \bar{A}(S) \cap R_j$, then $\mathbf{y}'' \in W^u(\mathbf{y}', R_j)$ for some $\mathbf{y}' \in \bar{A}(D)$ and $W^u(\mathbf{y}'', R_j) = W^u(\mathbf{y}', R_j) \subset \bar{A}(S) \cap R_j$. So $\bar{A}(S) \cap R_j$ is a u-subrectangle of R_j. □

Theorem 4.16. *For each $\mathbf{a} \in \Sigma_B$ the set $\bigcap_{j\in\mathbb{Z}} \bar{A}^{-j}R_{a_j}$ consists of a single point, denoted $\pi(\mathbf{a})$. The map $\pi : \Sigma_B \to \hat{\mathbb{T}}^2$ is a continuous surjection, $\pi \circ \sigma = \bar{A} \circ \pi$, and π is one-to-one over the residual set $Y = \hat{\mathbb{T}}^2 \setminus \bigcup_{j\in\mathbb{Z}} \bar{A}^j\left(\partial^s\mathcal{R} \cup \partial^u\mathcal{R}\right)$.*

Proof. If $a_1a_2\cdots a_n$ is a word with $B_{a_ja_{j+1}} = 1$, then inductively one sees that

$$\bigcap_{j=1}^{n} \bar{A}^{n-j}R_{a_j} = R_{a_n} \cap \bar{A}\left(\bigcap_{j=1}^{n-1} \bar{A}^{n-1-j}R_{a_j}\right)$$

is a u-subrectangle of R_{a_n}. From this one gets that

$$K_n(\mathbf{a}) = \bigcap_{j=-n}^{n} \bar{A}^{-j} R_{a_j}$$

is nonempty and the closure of its interior. As $K_n(\mathbf{a}) \supset K_{n+1}(\mathbf{a}) \supset \cdots$ we have

$$K(\mathbf{a}) = \bigcap_{j=-\infty}^{\infty} \bar{A}^{-j} R_{a_j} = \bigcap_{n=1}^{\infty} K_n(\mathbf{a}) \neq \emptyset$$

If $\mathbf{x}, \mathbf{y} \in K(\mathbf{a})$, then $\bar{A}^j \mathbf{x}, \bar{A}^j \mathbf{y} \in R_{a_j}$ are close for all $j \in \mathbb{Z}$ and so $\mathbf{x} = \mathbf{y}$ by the expansiveness of $\bar{A}$. As

$$\begin{aligned} K(\sigma \mathbf{a}) &= \bigcap_j \bar{A}^{-j} R_{a_{j+1}} = \bar{A} \left(\bigcap_j \bar{A}^{-j} R_{a_j} \right) \\ &= \bar{A} K(\mathbf{a}) \end{aligned}$$

one has $\pi \circ \sigma = \bar{A} \circ \pi$. That π is continuous is proved similarly to the earlier continuity lemma. As $\partial^s \mathcal{R} \cup \partial^u \mathcal{R}$ is nowhere dense, Y is residual. For $\mathbf{x} \in Y$ pick a_j with $\bar{A}^j \mathbf{x} \in R_{a_j}$. As $\mathbf{x} \in Y$, $\bar{A}^j \mathbf{x} \in \operatorname{int}\left(R_{a_j}\right)$ and so $B_{a_j a_{j+1}} = 1$. Thus $\mathbf{a} = \{a_j\} \in \Sigma_B$ and $\mathbf{x} = \pi(\mathbf{a})$. If $\mathbf{x} = \pi(\mathbf{b})$, then $\bar{A}^j \mathbf{x} \in R_{b_j}$ and $b_j = a_j$ because $\bar{A}^j \mathbf{x} \notin \partial^s \mathcal{R} \cup \partial^u \mathcal{R}$; so π is injective over Y. As $\pi\left(\Sigma_B\right)$ is a compact subset of $\hat{\mathbb{T}}^2$ containing a dense set Y, $\pi\left(\Sigma_B\right) = \hat{\mathbb{T}}^2$. □

Generalizations

References

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A Two symbolic examples

We now study the space $\mathcal{A}^{\mathbb{Z}} \times \mathcal{B}^{\mathbb{N}_0}$ mentioned in the introduction. Its first coordinate is a two-sided sequence; its second coordinate is a one-sided sequence. We give two compact invariant subsets. In the first, Hausdorff and Minkowski dimensions differ. In the second, these dimensions agree, but exceed the dimension of every invariant probability measure.

These are statements about the metric defined below. We do not identify these sets with subsets of the torus. In particular, neither example answers the dimension question for the matrix $\left(\begin{smallmatrix} 4 & 2 \\ 2 & 2 \end{smallmatrix}\right)$.

A.1 The space, the metric, and cylinder sets

Let $\mathcal{A}$ and $\mathcal{B}$ be finite alphabets, and put

$$\Sigma = \mathcal{A}^{\mathbb{Z}} \times \mathcal{B}^{\mathbb{N}_0}, \qquad \sigma(x, y) = ((x_{j+1})_{j\in\mathbb{Z}}, (y_{j+1})_{j\geq 0}).$$

For $z = (x, y)$ and $z' = (x', y')$, define

$$N(z, z') = \min(\{|j| : x_j \neq x'_j\} \cup \{j \geq 0 : y_j \neq y'_j\}), \qquad d(z, z') = 2^{-N(z,z')}.$$

The minimum of the empty set is ∞, and $2^{-\infty} = 0$. Thus the closest disagreement to index zero determines the distance. This metric induces the product topology. It is an ultrametric: if two pairs agree at every coordinate of weight greater than r, so does the remaining pair. Hence $d(z, z'') \leq \max\{d(z, z'), d(z', z'')\}$. The space Σ is compact and σ is continuous. Throughout this appendix, invariant means $\sigma(E) = E$.

For $n \geq 1$, the *level-n cylinder* containing z is

$$C_n(z) = \{(x', y') : x'_j = x_j \ (-n \leq j < n), \quad y'_j = y_j \ (0 \leq j < n)\}.$$

Its diameter is at most 2^{-n}. Conversely, a nonempty set of diameter strictly less than 2^{-n} lies in one level-n cylinder. For $E \subseteq \Sigma$, let $K_n(E)$ be the number of such cylinders meeting E. If $\mathcal{N}_r(E)$ denotes the least number of sets of diameter at most r covering E, then, for $n \geq 2$,

$$K_{n-1}(E) \leq \mathcal{N}_{2^{-n}}(E) \leq K_n(E). \tag{A.1}$$

Indeed, each set in the left-hand covering fixes all coordinates specified at level $n-1$. Covering by sets of diameter r or by balls of radius r gives the same dimension: either convention bounds the other after replacing r by $2r$. Consequently,

$$\underline{\dim}_{\mathrm{M}} E = \liminf_{n\to\infty} \frac{\log_2 K_n(E)}{n}, \qquad \overline{\dim}_{\mathrm{M}} E = \limsup_{n\to\infty} \frac{\log_2 K_n(E)}{n}.$$

Lemma A.1. *Let $E \subseteq \Sigma$ be Borel. Suppose a Borel probability ν with $\nu(E) = 1$ satisfies*

$$\nu(C_n(z)) \le C2^{-sn} \qquad (z \in \Sigma,\ n \ge 1)$$

for constants $s > 0$ and $C > 0$. Then every Borel set of full ν-measure has Hausdorff dimension at least s. In particular, $\dim_{\mathrm{H}} E \ge s$.

Proof. If $2^{-(n+1)} \le \operatorname{diam} U < 2^{-n}$, then U lies in one level-n cylinder. Its outer measure therefore satisfies

$$\nu^*(U) \le C2^{-sn} \le C2^s(\operatorname{diam} U)^s.$$

Singletons have measure zero, by letting $n \to \infty$ in the assumed bound. For any countable cover $F \subseteq \bigcup_i U_i$ of a full-measure set by sufficiently small sets,

$$1 \le \sum_i \nu^*(U_i) \le C2^s \sum_i (\operatorname{diam} U_i)^s.$$

Taking infima over these covers gives $\mathcal{H}^s(F) \ge (C2^s)^{-1} > 0$, and hence $\dim_{\mathrm{H}} F \ge s$. □

A.2 Different Hausdorff and Minkowski dimensions

Take $P = \{1, 2\}$, $Q = \{1, 2, 3, 4\}$, $\mathcal{A} = \{0\} \cup P$, and $\mathcal{B} = \{0\} \cup Q$. It is convenient to construct two-sided pairs first and then discard the negative coordinates of the second sequence. Write

$$\widehat{\Sigma} = \mathcal{A}^{\mathbb{Z}} \times \mathcal{B}^{\mathbb{Z}}, \qquad p(a, b) = (a, (b_j)_{j \ge 0}).$$

For each $n \ge 1$, put $g_n = \lceil \sqrt{n} \rceil$. Given $u \in P^n$ and $v \in Q^n$, define a pair of sequences by

$$a_j = \begin{cases} u_j & 0 \le j < n, \\ 0 & \text{otherwise}, \end{cases} \qquad b_j = \begin{cases} v_{j-n-g_n} & n + g_n \le j < 2n + g_n, \\ 0 & \text{otherwise}. \end{cases}$$

Here words are indexed from 0 to $n - 1$. Thus a block of n first-coordinate symbols is followed by g_n zero pairs, then by a block of n second-coordinate symbols. Let $\mathcal{F}$ consist of all these pairs and all their integer translates. Define

$$Y = \overline{\mathcal{F}} \subseteq \widehat{\Sigma}, \qquad Z = p(Y).$$

The closure is taken in the product topology.

Proposition A.2. *The set Z is compact, $\sigma(Z) = Z$, and*

$$\dim_{\mathrm{H}} Z = 2 < 3 = \underline{\dim}_{\mathrm{M}} Z = \overline{\dim}_{\mathrm{M}} Z.$$

Proof. The two-sided shift is a homeomorphism and permutes $\mathcal{F}$. It therefore maps Y onto itself. Since p is continuous and commutes with the shift, Z is compact and $\sigma(Z) = Z$.

Hausdorff dimension. The set $\mathcal{F}$, and hence $p(\mathcal{F})$, is countable. We describe the other limits in Y. If the block lengths in a convergent sequence from $\mathcal{F}$ remain bounded, then either a subsequence is constant or both blocks leave every fixed finite interval. The limit belongs to $\mathcal{F}$ or is zero.

Suppose instead that the block lengths tend to infinity. The distance between a nonzero first-coordinate symbol and a nonzero second-coordinate symbol is at least $g_n + 1$, which tends to infinity. A limit therefore cannot have both kinds of nonzero symbols at fixed indices. Also, the support of each coordinate is an interval: if two indices carry nonzero symbols, every index between them does too. This property persists under limits, since the alphabets are finite.

It follows that each remaining limit has only P symbols in its first coordinate, on an interval of $\mathbb{Z}$, or only Q symbols in its second coordinate, on an interval of $\mathbb{Z}$. All other symbols are zero. There are countably many intervals of $\mathbb{Z}$, including finite intervals, half-lines, $\mathbb{Z}$ itself, and the empty interval. For a fixed interval, the first type has at most 2^{2t} level-t cylinders after projection; the second has at most 4^t. Each family has upper Minkowski dimension at most 2, and thus Hausdorff dimension at most 2. Countable stability of Hausdorff dimension gives $\dim_{\mathrm{H}} Z \le 2$.

To obtain the reverse inequality, observe that $P^{\mathbb{Z}} \times \{0^{\mathbb{N}_0}\} \subseteq Z$. Indeed, center increasingly long first-coordinate blocks around zero and choose their symbols to match any prescribed sequence. The second block leaves every finite interval. On this full binary shift, the uniform product probability gives every nonempty level-t cylinder mass 2^{-2t}. Lemma A.1 gives $\dim_{\mathrm{H}} Z \ge 2$.

Minkowski dimension. Fix $t \ge 1$. A cylinder records the first coordinate on $[-t, t-1]$ and the second on $[0, t-1]$. For a fixed pair of visible support intervals, let u and v be their respective lengths. The first block precedes the second, so these intervals are disjoint as subsets of $[-t, t-1]$. Consequently,

$$u + v \le 2t, \qquad v \le t, \qquad 2^u 4^v = 2^{u+2v} \le 2^{3t}.$$

There are at most $C(t+1)^4$ pairs of intervals, for a constant C independent of t. The same bound includes the one-coordinate limits described above. Thus

$$K_t(Z) \le C(t+1)^4\, 2^{3t}. \tag{A.2}$$

For a lower bound, choose $n = t$ and translate the second block to $[0, t-1]$. The first block is then $[-t - g_t, -g_t - 1]$. The observed coordinates include $t - g_t$ freely chosen binary symbols and t freely chosen four-valued symbols. Therefore

$$K_t(Z) \ge 2^{t-g_t} 4^t = 2^{3t-g_t}. \tag{A.3}$$

Since $g_t/t \to 0$, equations (A.2)–(A.3) and (A.1) prove that both Minkowski dimensions are 3. □

The two conditions on the gap have different uses: $g_n \to \infty$ excludes limits containing both kinds of symbols, while $g_n/n \to 0$ gives the lower bound for the Minkowski dimension.

A.3 A set larger in dimension than all its invariant measures

For a Borel probability measure μ on a metric space, define

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} F : F \text{ is Borel and } \mu(F) = 1\}.$$

An invariant probability on E means a Borel probability satisfying $\mu(\sigma^{-1}F) = \mu(F)$ for every Borel $F \subseteq E$. For a compact invariant set E, put

$$D_{\mathrm{inv}}(E) = \sup\{\dim_{\mathrm{H}} \mu : \mu \text{ is an invariant probability on } E\}.$$

The supremum includes all invariant probabilities, without an ergodicity or independence assumption. Necessarily $D_{\mathrm{inv}}(E) \le \dim_{\mathrm{H}} E$. In the preceding example equality holds: the invariant binary product probability has dimension 2. We next make this inequality strict.

Fix an integer $m \ge 3$. Take $\mathcal{A} = \{0, 1, *\}$ and $\mathcal{B}_m = \{0, 1, \dots, m-1\}$, with the same metric d. Let $X_m \subseteq \mathcal{A}^{\mathbb{Z}} \times \mathcal{B}_m^{\mathbb{N}_0}$ consist of all pairs satisfying

$$x_j = * \implies x_{j+1} = * \quad (j \in \mathbb{Z}), \qquad x_j \ne * \implies y_j = 0 \quad (j \ge 0). \tag{A.4}$$

The first coordinate may change once from binary symbols to $*$, but cannot change back. The second coordinate is free only where the first coordinate is $*$.

These conditions are closed and are preserved by σ. Every $(x, y) \in X_m$ has a preimage in X_m: set $x'_j = x_{j-1}$, $y'_0 = 0$, and $y'_j = y_{j-1}$ for $j \ge 1$. Hence X_m is compact and $\sigma(X_m) = X_m$.

For $k \in \mathbb{Z}$, define the set with change at k by

$$E_k = \{(x, y) \in X_m : x_j \in \{0, 1\} \ (j < k), \quad x_j = * \ (j \ge k)\}.$$

The cases with no change form

$$P_0 = \{0, 1\}^{\mathbb{Z}} \times \{0^{\mathbb{N}_0}\}, \qquad Q_m = \{*^{\mathbb{Z}}\} \times \mathcal{B}_m^{\mathbb{N}_0}.$$

These are disjoint Borel sets and give the decomposition

$$X_m = P_0 \,\dot\cup\, Q_m \,\dot\cup\, \bigcup_{k \in \mathbb{Z}} E_k. \tag{A.5}$$

Proposition A.3. *For every integer $m \ge 3$,*

$$D_{\mathrm{inv}}(X_m) = \max\{2, \log_2 m\} < \dim_{\mathrm{H}} X_m = \underline{\dim}_{\mathrm{M}} X_m = \overline{\dim}_{\mathrm{M}} X_m = 1 + \log_2 m.$$

In particular, $D_{\mathrm{inv}}(X_4) = 2 < 3 = \dim_{\mathrm{H}} X_4 = \dim_{\mathrm{M}} X_4$.

Proof. On E_0, the binary coordinates x_j for $j < 0$ and the m-valued coordinates y_j for $j \ge 0$ are independent choices. There are exactly $2^n m^n$ level-n cylinders. Give these free coordinates their uniform product probability. Every nonempty cylinder then has mass $(2m)^{-n}$. Lemma A.1 yields

$$\dim_{\mathrm{H}} X_m \ge \dim_{\mathrm{H}} E_0 \ge 1 + \log_2 m.$$

Upper bound for the set dimensions. For $-n \le k \le 0$, the number of level-n cylinders meeting E_k is $2^{n+k}m^n$. For $0 \le k \le n$, it is $2^{n+k}m^{n-k}$. Both quantities are at most $(2m)^n$: in the second case their ratio to $(2m)^n$ is $(2/m)^k \le 1$. Changes to the left of the observed interval give at most m^n additional cylinders, while changes to its right give at most 2^{2n}; these counts also include Q_m and P_0, respectively. Thus

$$K_n(X_m) \le (2n+3)(2m)^n.$$

Equation (A.1) gives $\overline{\dim}_{\mathrm{M}} X_m \le 1 + \log_2 m$. Together with the Hausdorff lower bound, this proves equality of the three set dimensions.

Invariant measures. Let μ be any invariant Borel probability on X_m. The shift moves a change at $k+1$ to a change at k, so

$$\sigma^{-1}(E_k) = E_{k+1}, \qquad \mu(E_k) = \mu(E_{k+1}) \quad (k \in \mathbb{Z}).$$

Since the sets E_k are pairwise disjoint, their common mass is zero. By countable additivity and (A.5),

$$\mu\left(\bigcup_{k\in\mathbb{Z}} E_k\right) = 0, \qquad \mu(P_0 \cup Q_m) = 1.$$

The cylinder counts for P_0 and Q_m are 2^{2n} and m^n, respectively. Their uniform product probabilities have cylinder masses 2^{-2n} and m^{-n}. By (A.1) and Lemma A.1,

$$\dim_{\mathrm{H}} P_0 = 2, \qquad \dim_{\mathrm{H}} Q_m = \log_2 m.$$

The full-measure set $P_0 \cup Q_m$ therefore gives

$$\dim_{\mathrm{H}} \mu \le \dim_{\mathrm{H}}(P_0 \cup Q_m) = \max\{2, \log_2 m\}.$$

Both product probabilities are invariant. Lemma A.1 applies to every full-measure set for either probability, so their measure dimensions are 2 and $\log_2 m$. One of them attains the upper bound. Finally, $1 + \log_2 m$ is larger than both 2 and $\log_2 m$ when $m \ge 3$. □

A.4 Relation to the toral problem

The two examples establish different strict inequalities:

$$D_{\mathrm{inv}}(Z) = \dim_{\mathrm{H}} Z = 2 < \dim_{\mathrm{M}} Z = 3, \qquad D_{\mathrm{inv}}(X_4) = 2 < \dim_{\mathrm{H}} X_4 = \dim_{\mathrm{M}} X_4 = 3.$$

The first uses countably many finite configurations to increase the number of cylinders needed at small scales. The second uses sets E_k which have large Hausdorff dimension but zero mass for every invariant probability.

All dimensions here use d, not the Euclidean metric on $\mathbb{T}^2$. The value 3 is therefore consistent, but cannot be the dimension of a toral subset. A symbolic coding of the natural extension does not by itself provide a dimension-preserving map from either constructed set to the torus. No such realization has been proved here. The examples show what can happen in this mixed sequence space; they give neither a proof nor a counterexample for the corresponding toral question.